Set-Valued Var. Anal. manuscript No.
(will be inserted by the editor)

Convex SIP problems with finitely representable compact index sets: immobile indices and the properties of the auxiliary NLP problem

Preprint

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Received: date / Accepted: date

Abstract In the paper, we consider a problem of convex Semi-Infinite Programming with a compact index set defined by a finite number of nonlinear inequalities. While studying this problem, we apply the approach developed in our previous works and based on the notions of immobile indices, the corresponding immobility orders and the properties of a specially constructed auxiliary nonlinear problem. The main results of the paper consist in the formulation of sufficient optimality conditions for a feasible solution of the original SIP problem in terms of the optimality conditions for this solution in a specially constructed auxiliary nonlinear programming problem and in study of certain useful properties of this finite problem.

Keywords Semi-Infinite Programming (SIP) · Constraint Qualifications (CQ) · lower level problem · immobile index · immobility order · optimality conditions · Nonlinear Programming (NLP)

Mathematics Subject Classification (2000) 90C25 · 90C30 · 90C34

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1 Introduction

A Semi-Infinite Programming problem consists in minimization of a real-valued finite dimensional function subject to infinitely many constraints. First works on SIP appeared in 60s of XX century in the papers of Charnes A., Cooper W.W., and Kortanek K.O. devoted to Linear SIP (see [4,5]). The first numerical method for SIP models arising in applications was suggested in early 70s by Gustafson S.-A. and Kortanek K.O. ([9]). Since that time the interest to SIP is constantly growing due to many important applications, both theoretical and practical. The information about history of SIP, its theoretical and numerical aspects, and the references can be found in [8,10,20,24,26].

Generally, SIP problems are formulated in the form

$$\min_{x\in\mathbb{R}^n} c(x) \qquad (1)$$
$$\text{s.t.} \quad f(x,t) \leq 0 \ \ \forall\, t \in T,$$

where $c : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \times T \to \mathbb{R}$ are real valued continuous functions, $T \subset \mathbb{R}^p$ is a compact (infinite) index set. When, additionally, the set T depends on the decision variable x, one gets a problem of the generalized SIP, GSIP (see [11]). Sometimes (see e.g. [3]), the objective function and the functions defining the feasible set depend additionally on a so called *perturbation parameter*.

The main idea that lays in the basis of the most efficient approaches to the optimality conditions in SIP is to substitute the infinitely many constraints of a given SIP problem by a finite number of constraints getting thus an auxiliary Nonlinear Programming (NLP) problem. The finite number of constraints of this problem should be chosen in such a way that under certain additional conditions (for example, regularity conditions or *Constraint Qualifications (CQ)*), the optimality of a given feasible solution of the original SIP problem, can be verified using the optimality conditions in the auxiliary NLP problem. To test the optimality in the NLP problem, the widespread results of Mathematical Programming can be applied.

In the SIP literature, there exist two main approaches to the constraints set's substitution: discretization and reduction (see [10,20] and the references therein).

When the **discretization** approach is being applied, the infinite index set T is replaced by an appropriate finite subset. The following theorem (see [1]) justifies this approach.

Theorem 1 *Suppose that the convex and consistent SIP problem (1) satisfies the following condition:*

$$\textit{for any } n+1 \textit{ indices } t_j \in T,\ j = 1,\dots,n+1, \ \textit{there exists}$$
$$\textit{a point } \tilde{x} \in \mathbb{R}^n \textit{ such that } f(\tilde{x}, t_j) < 0, j = 1,\dots,n+1. \qquad (2)$$

Then there exists a finite set $\{t^0_1, t^0_2, \dots, t^0_n\} \subset T$, *such that* $val(1) = val(SIP_D)$, *where*

$$(SIP_D): \qquad \begin{array}{c} \min\ c(x), \\ s.t.\ \ f(x, t^0_j) \leq 0,\ \ j = 1,...,n. \end{array}$$

Here and in what follows, $val(P)$ denotes the optimal value of the objective function of an optimization problem (P).

Problem (SIP_D) is a discretization of the original SIP problem (1). Theorem 1 states that under condition (2) the test of optimality of $x^0 \in X$ in problem (1) can be reduced to the test of its optimality in the finite dimensional problem (SIP_D). It is also well known that condition (2) is essential in Theorem 1.

For problem (1) with finitely representable compact index set, the **reduction** approach is much more productive. The essence of the approach is as follows.

Under certain assumptions (see [10, 26] et al.) that should be satisfied for a feasible x^0, it is possible to prove that the set $\{\bar{t}_i, i \in J\} := \{t \in T : f(x^0, t) = 0\}$ is finite and there exists a neighborhood $U(x^0)$ of x^0 and continuous functions $t_i : U(x^0) \to T, i \in J$, such that $t_i(x^0) = \bar{t}_i, i \in J$, and for every $x \in U(x^0)$, the functions $t_i(x)$, $i \in J$, are all solutions of the parametric problem $\max\limits_{t \in T} f(x, t)$. In this case for $x \in U(x^0)$, infinitely many constraints $f(x, t) \le 0, t \in T$, can be locally replaced by a finite number of constraints $f(x, t_i(x)) \le 0, i \in J$. Therefore a feasible x^0 is a local optimal solution in the original SIP problem if and only if it is a local optimal solution in the following reduced problem:

$$(SIP_{red}) : \qquad \begin{array}{c} \min \ c(x), \\ \text{s.t.} \ \ f(x, t_i(x)) \le 0, \ i \in J. \end{array}$$

The reduction permits to extend various optimality (necessary or sufficient) conditions known from (finite) NLP, to (infinite) SIP problems.

Note that in practice, it is more easy to apply the discretization approach where the discretized problem (SIP_D) is formulated in terms of certain known constraint functions. But it is not a trivial task to make reduction since the reduced problem (SIP_{red}) contains the constraints that are formulated with the help of some functions $t_i(x), i \in J$, implicitly defined in the form of solutions of the optimization problem $\max\limits_{t \in T} f(x, t)$. In general case, to use these implicit functions and to apply the reduction approach, one needs to introduce some CQ. These CQs are less restrictive than (2) and the optimality conditions obtained using the reduction approach are more efficient since the approach uses specific structure and properties of the original SIP problem.

In the recent papers [3, 21], et al., the advanced techniques of variational analysis are applied to broad classes of infinite programming problems, including nonsmooth and nonconvex problems with arbitrary index sets. Under certain differentiability assumptions and the closedness CQ, new verifiable necessary optimality conditions are derived. Being applied to the convex SIP problems with compact index sets, this approach permits to obtain the subdifferential KKT(Karush-Kuhn-Tucker)-type optimality conditions under rather simple CQs.

In this paper, we study a rather general class of convex smooth SIP problems. The only condition that the index sets should satisfy is their compactness and finite representability, i.e. they have to be described by a finite number of inequalities. For a given SIP problem, we apply a kind of reduction approach that is based on a deep study of some optimal solutions of the lower level problem that form a special subset of the index set, the set of the immobile indices of the original SIP problem. The study of the properties of the immobile indices and their immobility orders permits us to construct an auxiliary (reduced) NLP problem of a special form. This problem possesses some important properties that can be efficiently explored when the optimality of the original SIP problem is studied. In particular,

• Our auxiliary problem is convex.

• In general, our auxiliary problem differs from (SIP_D) and (SIP_{red}) problems.

• Our auxiliary problem coincides with the problem (SIP_D) if the original SIP problem satisfies condition (2).

• Our auxiliary problem is formulated in terms of special functions that permit to formulate explicitly the optimality conditions for the lower level problem and possess other important properties.

• It is possible to prove that if a vector x^0 (that is feasible in problem (1)) is optimal in the reduced problem (SIP_{red}), then it is optimal in our auxiliary problem as well.

• Under assumptions, that are less restrictive than ones usually used for both discretization and reduction approaches, we can prove that a feasible x^0 is optimal in the original SIP problem if and only if it is optimal in our auxiliary problem.

The latest of the mentioned above properties, permits to conclude that the checking of the optimality of a feasible solution x^0 of the original SIP problem, can be substituted by the checking of its optimality in the auxiliary NLP problem. Therefore one can formulate the optimality conditions for SIP problem in terms of such conditions for the auxiliary NLP problem. The discovered properties of the NLP problem permit one to use the most efficient optimality conditions that should give new optimality conditions for SIP that differ from the known ones (see, e.g., [1, 8, 11, 12, 21, 23, 25]). Notice here that the assumptions we make for the convex SIP problem, are less restrictive than those that are usually made in the literature. Therefore, the new optimality conditions that one can obtain for the convex SIP using our approach are more general. We are going to devote our subsequent paper to formulation and proof of the new optimality criterion for the convex SIP problems with finitely representable index sets that can be obtained using our approach.

The rest of the paper is organized as follows. In section 2, we state the convex SIP problem with finitely representable index set, introduce basic notations and definitions for it, formulate and study the lower level problem. In section 3, we formulate new sufficient optimality conditions for the original convex SIP problem in terms of such conditions for a specially constructed auxiliary NLP problem. An illustrative example is presented. Section 4 is dedicated to study of the properties of this auxiliary problem. In particular, we show that it is convex and its inequality constraints satisfy the Slater type CQ. The conclusions and final remarks are made in the final section 5.

2 Convex SIP problem with finitely represented index set

2.1 Problem statement and the basic notions

Consider a SIP problem in the form (1), where $T \subset \mathbb{R}^p$ is a compact index set defined by a finite system of inequalities

$$T = \{t \in \mathbb{R}^p : \; g_s(t) \leq 0, \; s \in S\}, |S| < \infty. \tag{3}$$

Let us denote the obtained problem by (SIP):

$$(SIP): \qquad \min_{x \in \mathbb{R}^n} c(x), \qquad \text{s.t.} \quad f(x,t) \leq 0 \quad \forall \, t \in T.$$

In what follows, we assume that the objective function $c(x)$ and the constraint function $f(x,t)$, for all $t \in T$, are convex w.r.t. $x \in \mathbb{R}^n$, and hence the problem (SIP) is convex. Assume also that functions $c(x)$, $f(x,t)$ and $g_s(t)$ are sufficiently smooth w.r.t. $x \in \mathbb{R}^n$, and $t \in \mathbb{R}^p$. The sufficient smoothness means here that the partial derivatives of these functions of all orders that we will need in sequel, exist and are continuous for all respective variables.

Denote by X the set of feasible solutions in problem (SIP) (the feasible set):

$$X = \{x \in \mathbb{R}^n : f(x,t) \leq 0 \; \forall \; t \in T\},$$

and for a given $x \in X$, denote by $T_a(x)$ the corresponding set of the active indices (the active index set):

$$T_a(x) = \{t \in T : f(x,t) = 0\}.$$

Let us suppose that problem (SIP) is consistent, i.e., $X \neq \emptyset$.

In study of optimality both for NLP and SIP problems, it is common to impose certain additional conditions on their constraints (CQs or the *regularity conditions*). Without the additional conditions, the known optimality conditions may not hold true. Different CQs have been proposed in the literature for different classes of SIP problems (see [1,12,21] et al.).

In the papers ([15]-[17]), for a more restricted class of SIP problems (convex problems with polyhedral index sets), we defined the set of indices, active for all feasible x (immobile indices) and showed how these indices can be used to formulate and prove new optimality conditions that do not use any CQ, i.e. are CQ-free. In the present paper, we will generalize this approach for the convex SIP problems with index sets defined in (3). Notice that this generalization is not trivial and is based on a number of new notions and important theoretical results.

One of the basic concepts that we will use here is that of the immobile index.

Definition 1 An index $t \in T$ is said to be immobile in problem (SIP) if

$$f(x,t) = 0 \text{ for all } x \in X.$$

Denote by $T^* \subset T$ the set of *all* immobile indices in problem (SIP).

The immobile (or carrier) indices play an important role in study of feasibility and optimality in SIP (see [17,19] and the references there). It can be proved that the violation of almost all CQs implies $T^* \neq \emptyset$. The main aim of our study is to formulate and prove for problem (SIP) the new optimality conditions that do not require any CQ . Hence, in what follows, it is reasonable to suppose that $T^* \neq \emptyset$.

Notice that the condition $T^* \neq \emptyset$ is not restrictive in our study, i.e. the set of immobile indices may be empty as well. It easy to show that the emptiness of the set T^* implies the fulfillment of the Slater condition (see Corollary 3 below), and hence the constructions and statements presented below will be trivially fulfilled and our auxiliary NLP problem (see (23)) will coincide with the discritized problem (SIP_D).

2.2 The lower level problem and the regularity conditions

Usually in SIP literature, the study of optimality of a feasible solution $x \in X$ in the original SIP problem (1) is connected with a so-called *lower level problem* that in our case can be written in the form:

$$(LLP(x)): \qquad \max_t f(x,t), \qquad \text{s.t. } t \in T,$$

where the index set T is defined in (3).

Notice that this problem is a parametric NLP problem that possesses the following important properties:

a) the feasible set of $(LLP(x))$ coincides with the index set T of problem (SIP), and
b) each immobile index of problem (SIP) is a solution of $(LLP(x))$ for all $x \in X$.

Consider an index $t \in T$ that is feasible in the lower level problem $(LLP(x))$. For t, denote by $S_a(t) = \{s \in S : \ g_s(t) = 0\}$ the index set of active in t constraints of problem $(LLP(x))$.

Following usual notations (see e.g. [7] and [13]), for any $t \in T$, let us introduce the sets:

– *tangent cone* to the index set T at the point t:

$$D_1(t) = \{l \in \mathbb{R}^p : \ \exists \text{ sequences } \{t^k\}_{k=1}^{\infty}, t^k \in T, \text{ and } \{\theta_k\}_{k=1}^{\infty}, \theta_k \in \mathbb{R},$$

$$\text{such that } \lim_{k\to\infty} \theta_k = +0, \ t^k = t + \theta_k l + o(\theta_k)\};$$

– *linearized tangent cone* to the index set T at the point t:

$$L_1(t) = \{l \in \mathbb{R}^p : \ \frac{\partial g_s^T(t)}{\partial t} l \leq 0, \ s \in S_a(t)\};$$

– *second order tangent cone* to the index set T at the point t:

$$D_2(t) = \{(l, w) \in \mathbb{R}^{2p} : \exists \text{ sequences } \{t^k\}_{k=1}^{\infty}, \ t^k \in T, \text{ and } \{\theta_k\}_{k=1}^{\infty}, \theta_k \in \mathbb{R},$$
$$\text{such that } \lim_{k\to\infty} \theta_k = +0, \ t^k = t + \theta_k l + \frac{1}{2}\theta_k^2 w + o(\theta_k^2)\};$$

– *second order linearized tangent cone* to the set T at the point t:

$$L_2(t) = \{(l, w) \in \mathbb{R}^{2p} : l \in L_1(t), \ \tfrac{\partial g_s^T(t)}{\partial t} w + l^T \tfrac{\partial^2 g_s(t)}{\partial t^2} l \leq 0, \ s \in \tilde{S}_a(t, l)\}$$
$$\text{where } \tilde{S}_a(t, l) = \{s \in S_a(t) : \tfrac{\partial g_s^T(t)}{\partial t} l = 0\}.$$

It is easy to verify that for a given $t \in T$, the following relationships between the sets introduced above hold true:

$$D_1(t) \subset L_1(t), \ D_2(t) \subset L_2(t), \ D_2(t, 0) = D_1(t), \ L_2(t, 0) = L_1(t);$$
$$\text{if } (l, w) \in D_2(t), \text{ then } l \in D_1(t); \ \text{ if } (l, w) \in L_2(t), \text{ then } l \in L_1(t), \quad (4)$$

where $D_2(t, l) := \{w \in \mathbb{R}^p : \ (l, w) \in D_2(t)\}$, $L_2(t, l) := \{w \in \mathbb{R}^p : (l, w) \in L_2(t)\}$.

Following [7],[13] et al., let us make the following definition.

Definition 2 Given index $t \in T$, we say that in the lower level problem, the *first order Abadie-type* regularity condition is satisfied at t if $L_1(t) = D_1(t)$, and the *second order Abadie-type regularity condition* is satisfied at t if

$$L_2(t) \subset D_2(t), \text{ i.e. } L_2(t) = D_2(t). \quad (5)$$

Notice here that from (4) and (5), it follows that the *second order Abadie regularity condition* implies fulfillment of the *first order* one as well, i.e.

$$L_2(t) = D_2(t) \text{ for } t \in T \ \Rightarrow \ L_1(t) = D_1(t).$$

The Abadie regularity conditions may be considered as one of the weakest regularity conditions (CQs). Since these conditions are rather difficult to check in practice, many authors use in their studies other more strong regularity conditions that guarantee the fulfillment of the first and second order Abadie conditions and and can be more easily verified.

Consider the following CQs for the lower level problem ($LLP(x)$):

– the Mangasarian-Fomovitz CQ holds at $t \in T$ if

$$\exists l \in \mathbb{R}^p : \ \frac{\partial g_s^T(t)}{\partial t} l < 0, \ s \in S_a(t); \quad (MFCQ)$$

– the condition of *positive linear independency* of vectors $\frac{\partial g_s(t)}{\partial t}$, $s \in S_a(t)$, holds at $t \in T$ if

$$\sum_{s \in S_a(t)} y_s \frac{\partial g_s(t)}{\partial t} = 0, \; y_s \geq 0, \; s \in S_a(t) \; \Rightarrow y_s = 0, s \in S_a(t). \quad (PLICQ)$$

In [1] (see Proposition 5.47), it is proved that the conditions $(MFCQ)$ and $(PLICQ)$ are equivalent.

Let us now study a relationship between these CQs and the Abadie-type regularity conditions for the NLP problem $(LLP(x))$.

Proposition 1 *Given the set T defined in (3), suppose that the condition (MFCQ) holds at an index $t \in T$. Then the second order Abadie regularity condition (5) holds at t as well.*

Proof. Given $t \in T$, consider vector $(l, w) \in L_2(t)$. Then

$$l \in L_1(t), \; \frac{\partial g_s^T(t)}{\partial t} w + l^T \frac{\partial^2 g_s(t)}{\partial t^2} l \leq 0, \; s \in \tilde{S}_a(t, l). \tag{6}$$

By assumption, there exists a vector $l^0 \in \mathbb{R}^p$ such that

$$\frac{\partial g_s^T(t)}{\partial t} l^0 < 0, \; s \in S_a(t). \tag{7}$$

Consider a sequence $\{\theta_k\}_{k=1}^{\infty}$, $\theta_k \in \mathbb{R}$ such that $\lim_{k \to \infty} \theta_k = +0$. Construct a sequence of vectors $t^k, k = 1, 2, \ldots$ according to the following rule:

$$t^k = t + \theta_k l + \frac{1}{2}\theta_k^2 w + \frac{1}{6}\theta_k^3 \alpha l^0, k = 1, 2, \ldots,$$

where $\alpha > 0$ is some fixed constant that will be defined later. A general term of this sequence evidently admits the following representation:

$$t^k = t + \theta_k l + \frac{1}{2}\theta_k^2 w + o(\theta_k^2), k = 1, 2, \ldots. \tag{8}$$

Let us show that it is possible to choose $\alpha > 0$ in such a way that $t^k \in T$ for sufficiently large $k > 0$.

Given $s \in S_a(t)$, consider a function $\tilde{g}_s(\theta) := g_s(t + \theta l + \frac{1}{2}\theta^2 w + \frac{1}{6}\alpha l^0 \theta^3)$, $\theta \geq 0$. The following expansion of this function in the neighborhood of 0 can be done:

$$\tilde{g}_s(\theta) = \tilde{g}_s(0) + \theta \frac{d\tilde{g}_s(0)}{d\theta} + \frac{1}{2}\theta^2 \frac{d^2\tilde{g}_s(0)}{d\theta^2} + \frac{1}{6}\theta^3 \frac{d^3\tilde{g}_s(0)}{d\theta^3} + o(\theta^3). \tag{9}$$

Calculate:

$$\tilde{g}_s(0) = g_s(t) = 0; \quad \frac{d\tilde{g}_s(0)}{d\theta} = \frac{\partial g_s^T(t)}{\partial t} l; \frac{d^2\tilde{g}_s(0)}{d\theta^2} = \frac{\partial g_s^T(t)}{\partial t} w + l^T \frac{\partial^2 g_s(t)}{\partial t^2} l;$$

$$\frac{d^3\tilde{g}_s(0)}{d\theta^3} = \frac{\partial}{\partial t}(l^T\frac{\partial^2 g_s(t)}{\partial t^2}l)^T l + 3l^T\frac{\partial^2 g_s(t)}{\partial t^2}w + \alpha\frac{\partial g_s^T(t)}{\partial t}l^0,$$

and introduce the sets

$$S_1 := \{s \in S_a(t) : \frac{d\tilde{g}_s(0)}{d\theta} < 0\},\ S_2 := \{s \in S_a(t) \setminus S_1 :\ \frac{d^2\tilde{g}_s(0)}{d\theta^2} < 0\},$$

$$S_3 := S_a(t) \setminus (S_1 \cup S_2).$$

Set $\alpha := 1+\max\{0, -\gamma_s/(\frac{\partial g_s^T(t)}{\partial t}l^0),\ s \in S_3\}$, where $\gamma_s := \frac{\partial}{\partial t}(l^T\frac{\partial^2 g_s(t)}{\partial t^2}l)^T l + 3l^T\frac{\partial^2 g_s(t)}{\partial t^2}w$.

Then, by construction (see (6) and (7)), the following relations hold true:

$$\text{for } s \in S_1 : \frac{d\tilde{g}_s(0)}{d\theta} < 0; \quad \text{for } s \in S_2 : \frac{d\tilde{g}_s(0)}{d\theta} = 0,\ \frac{d^2\tilde{g}_s(0)}{d\theta^2} < 0;$$

$$\text{and for } s \in S_3 : \frac{d\tilde{g}_s(0)}{d\theta} = \frac{d^2\tilde{g}_s(0)}{d\theta^2} = 0,\ \frac{d^3\tilde{g}_s(0)}{d\theta^3} < 0.$$

Taking into account these relations, the expansion (9), and the inequalities $g_s(t) < 0,\ s \in S \setminus S_a(t)$, we get $\tilde{g}_s(\theta_k) = g_s(t^k) < 0,\ s \in S$, that means that $t^k \in T$ for sufficiently large $k > 0$. Taking into account (8) and the definition of the set $D_2(t)$, we conclude that $(l, w) \in D_2(\bar{t})$ and hence $L_2(t) = D_2(t)$. □

2.3 Necessary optimality conditions for the lower level problem

Suppose that an immobile index $t \in T^*$ satisfies condition $(MFCQ)$. Then from the necessary optimality conditions for $t \in T^*$ in the lower level problems $(LLP(x))$, it follows that for any $x \in X$, the following relations are satisfied:

– *the first order necessary optimality conditions:* $\exists y_s = y_s(x), s \in S_a(t),$ such that

$$\frac{\partial f(x,t)}{\partial t} = \sum_{s\in S_a(t)} y_s\frac{\partial g_s(t)}{\partial t},\ y_s \geq 0,\ s \in S_a(t); \tag{10}$$

– *the second order necessary optimality conditions:*

$$\max_{y\in\Lambda(x,t)} l^T(-\frac{\partial^2 f(x,t)}{\partial t^2} + \sum_{s\in S_a(t)} y_s\frac{\partial^2 g_s(t)}{\partial t^2})l \geq 0 \text{ for all } l \in C(x,t), \tag{11}$$

where $\Lambda(x,t)$ is the set of all vectors $y = (y_s, s \in S_a(t))$ satisfying condition (10) and

$$C(x,t) = \{l \in L_1(t) :\ \frac{\partial f^T(x,t)}{\partial t}l = 0\} \tag{12}$$

is the *cone of critical directions* at the point t in problem $(LLP(x))$.

Let $t \in T^*,\ x \in \mathbb{R}^n,\ l \in\ C(x,t)$. Consider a parametric *Linear Programming (LP)* problem

$$(LP(x,t,l)): \qquad \max_{w} \frac{\partial f^T(x,t)}{\partial t} w$$
$$\text{s.t.} \quad \frac{\partial g_s^T(t)}{\partial t} w \le -l^T \frac{\partial^2 g_s(t)}{\partial t^2} l, \quad s \in S_a(t).$$

The dual problem to $(LP(x,t,l))$ has the form

$$(DLP(x,t,l)): \qquad \min_{y_s, s}(-\sum_{s \in S_a(t)} y_s l^T \frac{\partial^2 g_s(t)}{\partial t^2} l)$$
$$\text{s.t.} \quad \sum_{s \in S_a(t)} y_s \frac{\partial g_s(t)}{\partial t} = \frac{\partial f(x,t)}{\partial t},\ y_s \ge 0,\ s \in S_a(t).$$

If $x \in X$ and $(MFCQ)$ is satisfied at $t \in T^*$, then both linear problems, $(LP(x,t,l))$ and $(DLP(x,t,l))$, admit feasible solutions and hence have optimal solutions. Moreover, if vector $w(x,t,l)$ is an optimal solution of problem $(LP(x,t,l))$ and vector $(y_s(x,t,l), s \in S_a(t))$ is an optimal solution of problem $(DLP(x,t,l))$, then

$$\frac{\partial f^T(x,t)}{\partial t} w \le \frac{\partial f^T(x,t)}{\partial t} w(x,t,l) = \tag{13}$$
$$-\sum_{s \in S_a(t)} y_s(x,t,l) l^T \frac{\partial^2 g_s(t)}{\partial t^2} l \le -\sum_{s \in S_a(t)} y_s l^T \frac{\partial^2 g_s(t)}{\partial t^2} l,$$

for all $w \in L_2(t,l)$ and all $(y_s, s \in S_a(t)) \in \Lambda(x,t)$.

Consider the following auxiliary functions:

$$F_1(x,t,l) := \frac{\partial f^T(x,t)}{\partial t} l, \tag{14}$$
$$F_2(x,t,l) := l^T \frac{\partial^2 f(x,t)}{\partial t^2} l + val(DLP(x,t,l)) = l^T \frac{\partial^2 f(x,t)}{\partial t^2} l + val(LP(x,t,l)).$$

Given $x \in X$, the necessary optimality conditions (10) and (11) for $t \in T^*$ in problem $(LLP(x))$ can be formulated in terms of functions (14) as follows:

$$F_1(x,t,l) \le 0 \ \ \forall l \in L_1(t), \tag{15a}$$
$$F_2(x,t,l) \le 0 \ \ \forall l \in C(x,t). \tag{15b}$$

Notice that inequalities (15a), (15b) should be satisfied by all $x \in X$.

2.4 Immobility orders of the immobile indices along the tangencial directions

Taking into account conditions (15a) and (15b) that should be fulfilled for all $x \in X$ and for all immobile indices, we can give the following definition.

Definition 3 Given $t \in T^*$ satisfying $(MFCQ)$ and $l \in L_1(t),\ l \neq 0$, define the immobility order $q(t,l)$ of t along the direction l as follows:
- $q(t,l) = 0$ if $\exists \bar{x} = x(t,l) \in X$ such that $F_1(\bar{x},t,l) < 0$;
- $q(t,l) = 1$ if $F_1(x,t,l) = 0, \forall x \in X$ and $\exists \bar{x} = x(t,l) \in X$ such that $F_2(\bar{x},t,l) < 0$;
- $q(t,l) > 1$ if $F_1(x,t,l) = 0,\ F_2(x,t,l) = 0,\ \forall x \in X$.

Due to its polyhedral structure, the set $L_1(t)$ (the linearized tangent cone to the index set T at a point t) admits the following representation in terms of its extremal rays

$$L_1(t) = \{l \in \mathbb{R}^p :\ l = \sum_{i \in P(t)} \beta_i b_i(t) + \sum_{i \in I(t)} \alpha_i a_i(t),\ \alpha_i \geq 0,\ i \in I(t)\}, \quad (16)$$

where $b_i(t), i \in P(t)$, $|P(t)| < \infty$, are bidirectional extremal rays and $a_i(t), i \in I(t)$, $|I(t)| < \infty$, are unidirectional extremal rays of the cone $L_1(t)$. These extremal rays can be constructed using the procedure described in [6,18].

Let $t \in T^*$ be an immobile index. In the set $L_1(t)$, let us consider the subsets of directions l with immobility orders $q(t,l) = 0$ and $q(t,l) \geq 1$ respectively, and give a constructive description of these subsets in terms of the extremal rays.

It is evident that immobility order $q(t, b_i(t))$ along any bidirectional extremal ray $b_i(t)$, $i \in P(t)$, is greater than zero. The unidirectional extremal rays of the cone $L_1(t)$ can be divided into two groups: rays $a_i(t)$ with immobility order $q(t, a_i(t)) = 0$ and rays $a_i(t)$ with immobility order $q(t, a_i(t)) \geq 1$. In this connection let us denote

$$\begin{aligned} I_*(t) &:= \{i \in I(t) :\ q(t, a_i(t)) = 0\}, \\ I_0(t) &:= I(t) \setminus I_*(t) = \{i \in I(t) :\ q(t, a_i(t)) \geq 1\}. \end{aligned} \quad (17)$$

It is easy to show that the set $C_0(t) \setminus 0$, where

$$C_0(t) := \{l \in \mathbb{R}^p :\ l = \sum_{i \in P(t)} \beta_i b_i(t) + \sum_{i \in I_0(t)} \alpha_i a_i(t),\ \alpha_i \geq 0,\ i \in I_0(t)\},$$

describes the set of all directions $l \in L_1(t)$ with immobility orders $q(t,l) \geq 1$, i.e.

$$F_1(x,t,l) := \frac{\partial f^T(x,t)}{\partial t} l = 0,\ \forall l \in C_0(t),\ \forall x \in X. \quad (18)$$

Notice that it follows from (12) and (18), that

$$C_0(t) \subset C(x,t) \subset L_1(t) \ \ \forall x \in X. \quad (19)$$

It is evident that $q(t,l) = 0,\ \forall l \in L_1(t) \setminus C_0(t)$, hence for each $l \in L_1(t) \setminus C_0(t)$ there exists $x(t,l) \in X$ such that

$$F_1(x(t,l),t,l) := \frac{\partial f^T(x(t,l),t)}{\partial t} l < 0.$$

Given $t \in T^*$, with respect to the parametric representation of the cone $C_0(t)$ in terms of the extremal rays, the equalities (18), and the latter inequality, relations (15a) can be rewritten in the form

$$\begin{gathered} F_1(x,t,a_i(t)) = 0, i \in I_0(t), \quad F_1(x,t,b_i(t)) = 0, i \in P(t), \\ F_1(x,t,a_i(t)) \leq 0, i \in I_*(t), \ \forall x \in X. \end{gathered} \tag{20a}$$

According to (15b) and (19), we get for $t \in T^*$

$$F_2(x,t,l) = l^T \frac{\partial^2 f(x,t)}{\partial t^2} l + val(DLP(x,t,l)) \leq 0, \ \forall l \in C_0(t), \forall x \in X. \tag{20b}$$

3 Sufficient optimality conditions for problem (SIP)

In what follows, let us suppose that the following assumption is satisfied.

Assumption 1 *In the lower level problem ($LLP(x)$), the regularity condition (MFCQ) holds at any immobile index $t \in T^* \subset T$.*

We consider that the assumption is trivially fulfilled if $T^* = \emptyset$.

Theorem 2 *Let Assumption 1 be fulfilled. If for $x^0 \in X$, there exist subsets of indices $\bar{T} \subset T^*$, $T^0 \subset T_a(x^0) \setminus \bar{T}$, and subsets of vectors*

$$\{l_k(t), k = 1, ..., m(t)\} \subset \{l \in C_0(t) : F_2(x^0,t,l) = 0\}, t \in \bar{T}, \tag{21}$$

$$\textit{with } |\bar{T}| + |T^0| + \sum_{t \in \bar{T}} m(t) < \infty, \tag{22}$$

such that the vector x^0 is optimal in the following NLP problem:

$$\begin{gathered} \min \ c(x), \\ \text{s.t. } f(x,t) = 0, \ F_1(x,t,b_i(t)) = 0, i \in P(t), \ F_1(x,t,a_i(t)) = 0, i \in I_0(t), \\ F_1(x,t,a_i(t)) \leq 0, i \in I_*(t), F_2(x,t,l_k(t)) \leq 0, k = 1, ..., m(t), t \in \bar{T}; \\ f(x,t) \leq 0, t \in T^0, \end{gathered} \tag{23}$$

then x^0 is an optimal solution in problem (SIP).

Here and in what follows we suppose that $\{1, ..., m\} = \emptyset$ if $m = 0$.

Proof. Basing on the definitions of immobile indices and the results of the previous sections (in particular, taking into account formulas (20a) and (20b)), we can conclude that $X \subset \mathcal{X}$ where

$$\begin{aligned} \mathcal{X} = \big\{ & x \in \mathbb{R}^n : f(x,t) = 0, \ F_1(x,t,b_i(t)) = 0, i \in P(t), \\ & F_1(x,t,a_i(t)) = 0, i \in I_0(t), F_1(x,t,a_i(t)) \leq 0, i \in I_*(t), \\ & F_2(x,t,l) \leq 0, l \in C_0(t), \ t \in T^*; f(x,t) \leq 0, t \in T_a(x^0) \setminus T^* \big\}. \end{aligned}$$

It is evident that $\mathcal{X} \subset \bar{\mathcal{X}}$, where $\bar{\mathcal{X}}$ is the set of feasible solutions in problem (23). Consequently, if $x^0 \in X$ is an optimal solution in problem (23), it is optimal in problem (SIP) as well. $\square$

Remark 1 Notice that in Theorem 2 we do not require the fulfillment of Mangasarian-Fromovitz CQ at x^0 in the original SIP problem, i.e. we do not require the existence of $\xi \in \mathbb{R}^n$ such that $\quad \xi^T \frac{\partial f(x^0,t)}{\partial x} < 0 \;\; \forall t \in T_a(x^0)$.

Remark 2 Till now we have not used the assumption that the constraint function $f(x,t)$ is convex w.r.t. $x \in \mathbb{R}^n$ for all $t \in T$. Hence Theorem 2 holds true without the convexity assumption.

The inclusions $X \subset \mathcal{X} \subset \bar{\mathcal{X}}$, may give a false impression that the set $\bar{\mathcal{X}}$ approximates the feasible set X very roughly and that the sufficient optimality conditions formulated in Theorem 2 are not efficient. The following considerations justify that this is not true and the sufficient optimality conditions formulated in Theorem 2 are efficient ones.

• We will prove (in the subsequent paper) that for the convex SIP problem under an additional assumption (see Assumption 2 below) Theorem 2 holds true in the necessary part as well with

$$\bar{T} = T^*, \; |T^*| < \infty, \; |T^0| + \sum_{t \in \bar{T}} m(t) \leq n. \tag{24}$$

• We prove (see Appendix) that in the case of the convex SIP problem and $|T_a(x^0)| < \infty$ if the sufficient optimality conditions from [11] are fulfilled for $x^0 \in X$, then there exist finite subsets $\bar{T} \subset T^*$ and $\bar{I}(t) \subset I_0(t)$, $t \in \bar{T}$, $T^0 \subset T_a(x^0) \setminus T^*$, $|T^0| \leq n$, such that x^0 is optimal in the following NLP problem

$$\begin{aligned} &\min \; c(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (25)\\ \text{s.t. } \; & f(x,t) = 0, \; F_1(x,t,b_i(t)) = 0, i \in P(t), \; F_1(x,t,a_i(t)) = 0, i \in \bar{I}(t), \\ & F_1(x,t,a_i(t)) \leq 0, i \in I(t) \setminus \bar{I}(t), t \in \bar{T}; \; f(x,t) \leq 0, t \in T^0 \cup (T^* \setminus \bar{T}). \end{aligned}$$

The feasible set of problem (23) is contained in the feasible set of problem (25). Hence the optimality of $x^0 \in X$ in problem (25) implies its optimality in problem (23). Therefore if $x^0 \in X$ satisfies the sufficient optimality conditions from [11] then it satisfies the optimality conditions from Theorem 2. The example presented below shows that the converse is false: the optimality of feasible $x^0 \in X$ in problem (25) does not imply the fulfillment of the sufficient optimality conditions from [11]. Hence, in the case under consideration, the optimality conditions from Theorem 2 are more efficient than ones from [11].

Notice that among the sufficient optimality conditions that can be applied for the problem (SIP) considered in the paper (see, for example, [1,11,12,14, 22,23,25]), conditions from [11] are the strongest ones.

Example. Let us give an example that illustrates the introduced notations and applicability of Theorem 2. In this example, for a given feasible solution x^0 of the convex problem (SIP), the corresponding active indices satisfy the strong second order sufficient optimality conditions for the lower level problem, i.e., the conditions that permit to apply to the given SIP problem many of the known optimality conditions. We will show that in this example, the given feasible solution does not satisfy the known sufficient optimality conditions (we use here the results from [11], but nevertheless it is optimal since the conditions of Theorem 2 are satisfied.

Let us start with the following observation. It is easy to verify that the results of the paper can easily be generalized to the case of more than one semi-infinite constraint, i.e. for the convex SIP problems whose constraints have the form $f_i(x,t) \le 0,\ \forall t \in T_i,\ \ i = 1, \dots, r$, where for any $i = 1, \dots, r$, the index set T_i is finitely representable.

Let $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4,\ t = (\tau_1, \tau_2)^T \in \mathbb{R}^2$, and

$$\begin{aligned}
&f_1(x,t) = -\tau_1^2 x_1 + \tau_1\tau_2 x_1 + \tau_1 x_2 + (\sin\tau_1)x_3 + \tau_1 x_4 - \tau_2^2,\\
&f_2(x,t) = \tau_2 x_1 + (\tau_2+1)^2 x_2 + (1-\tau_2)x_3 + x_4 - (\tau_1-3)^2 + (\tau_1-3)\tau_2;\\
&T_1 = \{t \in \mathbb{R}^2 : -(\tau_1+1)^2 - (\tau_2-1)^2 \le -2, -0.5 \le \tau_1 \le 1, -0.5 \le \tau_2 \le 1\},\\
&T_2 = \{t \in \mathbb{R}^2 : (\tau_1-2.5)^2 + (\tau_2-0.5)^2 \le 0.5\}.
\end{aligned} \tag{26}$$

Notice that the index set T_1 is not convex here.

Consider the following SIP problem:

$$\begin{gathered}
\min\ (-x_2 + x_3)\\
\text{s.t. } f_1(x,t) \le 0,\ \forall t \in T_1,\ \ f_2(x,t) \le 0,\ \forall t \in T_2.
\end{gathered} \tag{27}$$

Problem (27) admits a feasible solution $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)^T$ such that

$$\begin{gathered}
x_1^0 = 2\sin(1) \approx 1.6829,\\
x_3^0 = (-(x_1^0)^2/4 + x_1^0)/(\sin(1)-1) = \frac{\sin(1)(2-\sin(1))}{\sin(1)-1} \approx -6.149464,\\
x_2^0 = 0.5(x_3^0 - x_1^0) \approx -3.9162,\ x_4^0 = -x_2^0 - x_3^0 \approx 10.0657.
\end{gathered} \tag{28}$$

Let us test the optimality of x^0 in problem (27), using the approach suggested in the paper.

Denote $t_1 := (0,0)^T \in T_1,\ t_2 := (3,0)^T \in T_2,\ t_3 := (1, x_1^0/2)^T \in T_1$. It is easy to verify that $f_1(x^0,t_1) = f_2(x^0,t_2) = f_1(x^0,t_3) = 0$ and the indices t_1, t_2, and t_3 form the active index set in x^0: $T_a(x^0) = \{t_1, t_2, t_3\}$. There are two immobile indices in problem (27): t_1 and t_2, hence $T^* = \{t_1, t_2\}$. It is easy to verify that

$$F_2(x^0, t_j, l) = l^T \frac{\partial^2 f_j(x^0, t_j)}{\partial t^2} l < 0,\ \forall l \in \mathbb{R}^2 \setminus \{0\}, j = 1, 2. \tag{29}$$

For the immobile index t_1, we have $S_a(t_1) = \{1\}$, and $L_1(t_1) = \{l \in \mathbb{R}^2 : -l_1 + l_2 \le 0\}$. According to the rules of constructing the extreme rays (see [18]), the cone $L_1(t_1)$ can be represented by one bidirectional ray $b_1(t_1) = (1,1)^T$ and one unidirectional ray $a_1(t_1) = (1,-1)^T$. It is easy to verify that $q(t_1, b_1(t_1)) = 1$ and $q(t_1, a_1(t_1)) = 1$. Then the sets in (17) are defined as follows: $I_*(t_1) = \emptyset,\ I_0(t_1) = \{1\}$.

Now consider the immobile index $t_2 = (3,0)^T \in T^*$. For this index, we have $S_a(t_2) = \{1\}$, and $L_1(t_2) = \{l \in \mathbb{R}^2 : l_1 - l_2 \le 0\}$. We can easily verify that the cone $L_1(t_2)$ is represented by one bidirectional ray $b_1(t_2) = (1,1)^T$ and one unidirectional ray $a_1(t_2) = (-1,1)^T$; and that the immobility orders of t_2 along these extreme rays are: $q(t_2, b_1(t_2)) = 1$; $q(t_2, a_1(t_2)) = 1$. Hence the sets in (17) are given by $I_*(t_2) = \emptyset$, $I_0(t_2) = \{1\}$.

It is easy to check that Assumption 1 is satisfied. Let us set $\bar{T} = T^*$ and $T^0 = T_a(x^0) \setminus T^* = \{t_3\}$ and consider the NLP problem (23). It follows from (29) that $m(t) = 0, t \in T^*$, hence problem (23) takes the form

$$\min\ (-x_2 + x_3)$$
$$\text{s.t.}\quad f_j(x,t_j) = 0, \frac{\partial f_j^T(x,t_j)}{\partial t} b_1(t_j) = 0, \frac{\partial f_j^T(x,t_j)}{\partial t} a_1(t_j) = 0,\ j = 1,2;$$
$$f_1(x,t_3) \le 0.$$

This problem can be rewritten in the form

$$\begin{gathered} \min\ (-x_2 + x_3) \\ \text{s.t.}\qquad x_2 + x_3 + x_4 = 0,\ \ x_1 + 2x_2 - x_3 = 0, \\ x_1(0.5x_1^0 - 1) + x_2 + \sin(1)x_3 + x_4 - 0.25(x_1^0)^2 \le 0. \end{gathered} \tag{30}$$

According to Theorem 2, the vector x^0 given in (28), is optimal in the SIP problem (27) if it is optimal in the LP problem (30). Let us show that the vector x^0 is optimal in this problem. Indeed, according to the optimality criterion for LP, the feasible vector x^0 is optimal in problem (30) iff there exist numbers $\lambda_1, \lambda_2, \lambda_3$, such that

$$\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} x_1^0/2 - 1 \\ 1 \\ \sin(1) \\ 1 \end{pmatrix} = 0,\quad \lambda_3 \ge 0.$$

The last conditions are satisfied with

$$\lambda_1 = -\lambda_3 \approx -3.1540, \lambda_2 = 0.5, \lambda_3 = -0.5/(\sin(1) - 1) \approx 3.1540 > 0.$$

Hence we have proved that the given vector x^0 is optimal in the LP problem (30). Consequently it is optimal in the SIP problem (27).

Now let us show that for the given vector x^0 in problem (27), many of sufficient optimality conditions known from the literature do not work. Among the sufficient optimality conditions [11,14,22,23,25] that can be applied for the problem (SIP) considered in the paper, we have chosen the strongest conditions from [11] (see Theorem 3 in the Appendix). Let us show that these conditions are not fulfilled for the given vector x^0 in problem (27).

In our example, relations (72) are as follows:

$$\bar{\lambda}_0 \frac{\partial c(x^0)}{\partial x} + \bar{\lambda}_1 \frac{\partial f_1(x^0,t_1)}{\partial x} + \bar{\lambda}_2 \frac{\partial f_2(x^0,t_2)}{\partial x} + \bar{\lambda}_3 \frac{\partial f_1(x^0,t_3)}{\partial x} = 0,\ \bar{\lambda}_i \ge 0, i = 0, ..., 3,$$

or equivalently

$$\bar{\lambda}_0 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \bar{\lambda}_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \bar{\lambda}_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \bar{\lambda}_3 \begin{pmatrix} 0.5x_1^0 - 1 \\ 1 \\ \sin(1) \\ 1 \end{pmatrix} = 0,\ \bar{\lambda}_i \geq 0,\ i = 0, ..., 3.$$

Since the system above admits a solution $\bar{\lambda}_0 = 0,\ \bar{\lambda}_1 \geq 0,\ \bar{\lambda}_2 = 0,\ \bar{\lambda}_3 = 0$, relations (73) take the form

$$\bar{\lambda}_1 \Big[\eta^T(\xi) \frac{\partial^2 f_1(x^0, t_1)}{\partial t^2} \eta(\xi) + 2\xi^T \frac{\partial^2 f_1(x^0, t_1)}{\partial x \partial t} \eta(\xi) \Big] > 0 \ \ \forall\ \xi \in \mathcal{K}, \xi \neq 0, \quad (31)$$

where $\eta(\xi) \in \mathbb{R}^2$ is a solution to the problem (see (74)):

$$\max \left(\frac{1}{2} \eta^T \frac{\partial^2 f_1(x^0, t_1)}{\partial t^2} \eta + \xi^T \frac{\partial^2 f_1(x^0, t_1)}{\partial x \partial t} \eta \right),\ \text{s.t.}\ (-1, 1)\eta \leq 0, \quad (32)$$

and $\mathcal{K} = \{\xi \in \mathbb{R}^4 :\ \xi^T \frac{\partial c(x^0)}{\partial x} \leq 0,\ \xi^T \frac{\partial f_1(x^0,\ t_1)}{\partial x} \leq 0,\ \xi^T \frac{\partial f_2(x^0,\ t_2)}{\partial x} \leq 0,\ \xi^T \frac{\partial f_1(x^0,\ t_3)}{\partial x} \leq 0\} = \{\xi \in \mathbb{R}^4 : -\xi_2 + \xi_3 \leq 0, \xi_2 + \xi_3 + \xi_4 \leq 0, (0.5x_1^0 - 1)\xi_1 + \xi_2 + \sin(1)\xi_3 + \xi_4 \leq 0\}$.

It is evident that $\bar{\xi} = (0, 1, 0, -1)^T \in \mathcal{K}$ and $\bar{\xi}^T \frac{\partial^2 f_1(x^0, t_1)}{\partial x \partial t} = (0, 0)$. Taking into account the last equality and relations (29), we conclude that problem (32) has the solution $\eta(\bar{\xi}) = 0$. Consequently, conditions (31) are not fulfilled for x^0. In other words, the optimality conditions from [11] are not able to recognize the optimality of the feasible vector x^0 in the convex SIP problem (27). It is easy to verify that the sufficient optimality conditions from [14, 22] are not fulfilled for x^0 in problem (27) as well. Remind once again that the given vector x^0 satisfies the optimality conditions formulated in Theorem 2. Hence for the given feasible solution x^0 of the convex SIP problem considered in the example, the sufficient optimality conditions proposed in the paper work “better” than the optimality conditions known in the literature.

4 The auxiliary NLP problem and its properties

The remainder of the paper is dedicated to study of the properties of the NLP problem (23). These properties will be used later to reformulate Theorem 2 in the form of the necessary optimality conditions, and to prove new explicit optimality conditions for convex SIP.

In what follows, we suppose that for all $t \in T$, the functions $f(x, t)$ are convex w.r.t. $x \in \mathbb{R}^n$.

4.1 The properties of the auxiliary functions (14)

First of all, recall that problem (23) is formulated in terms of the auxiliary functions (14). Hence to prove explicit optimality conditions for it, it is useful to state some important properties of these functions.

Lemma 1 *Suppose that (MFCQ) holds at $t \in T$ and that $\tilde{X} \subset \mathbb{R}^n$ is an arbitrary convex set such that $f(x,t) = 0,\ \forall x \in \tilde{X}$. Then for all $l \in L_1(t)$, the function $F_1(x,t,l)$ is convex w.r.t. x in $\tilde{X}$.*

Proof. Suppose that $l \in L_1(t)$ and $(MFCQ)$ holds at $t \in T^*$. From Proposition 1, we conclude that there exist sequences $\{t^k\}_{k=1}^{\infty},\ t^k \in T$, and $\{\theta_k\}_{k=1}^{\infty}, \lim\limits_{k\to\infty} \theta_k = +0$, such that

$$t^k = t + \theta_k l + o(\theta_k). \tag{33}$$

Let $x^1,\ x^2 \in \tilde{X}$. Set $x(\lambda) = \lambda x^1 + (1-\lambda)x^2$ for $\lambda \in [0,1]$. Since the set $\tilde{X}$ is convex, we have $x(\lambda) \in \tilde{X}$. For any $t^k \in T$, taking into account the convexity of function $f(x,t^k), x \in \tilde{X}$, we get

$$f(x(\lambda),t^k) \le \lambda f(x^1,t^k) + (1-\lambda) f(x^2,t^k),\ \forall \lambda \in [0,1]. \tag{34}$$

Since $f(x,t) = 0,\ \forall x \in \tilde{X}$, then from (33) and (34), it follows that for all $\lambda \in [0,1]$ and all $x^1, x^2 \in \tilde{X}$, the following inequality takes place:

$$\frac{\partial f^T(x(\lambda),t)}{\partial t} l + O(\theta_k) \le \lambda \frac{\partial f^T(x^1,t)}{\partial t} l + (1-\lambda)\frac{\partial f^T(x^2,t)}{\partial t} l + O(\theta_k).$$

Hence $\frac{\partial f^T(x(\lambda),t)}{\partial t} l \le \lambda \frac{\partial f^T(x^1,t)}{\partial t} l + (1-\lambda)\frac{\partial f^T(x^2,t)}{\partial t} l$, and taking into account (14), the function $F_1(x,t,l)$ is convex w.r.t. x in $\tilde{X}$ for all $l \in L_1(t)$. □

Lemma 2 *Suppose that (MFCQ) holds at $t \in T$ and that $\tilde{X} \subset \mathbb{R}^n$ is an arbitrary convex set such that*

$$f(x,t) = 0, \tag{35}$$
$$\frac{\partial f^T(x,t)}{\partial t} l \le 0,\ \forall l \in L_1(t),\ \frac{\partial f^T(x,t)}{\partial t} l = 0,\ \forall l \in L(t), \forall x \in \tilde{X},$$

where $L(t)$ is some subset of $L_1(t)$. Then for all $l \in L(t)$, the function $F_2(x,t,l)$ is convex w.r.t. x in $\tilde{X}$.

Proof. Suppose that $l \in L(t)$ and $(MFCQ)$ holds at $t \in T$. Consider a vector $w \in L_2(t,l)$. Hence $(l,w) \in L_2(t)$ and from Proposition 1 we have $(l,w) \in D_2(t)$ that implies the existence of the sequences $\{t^k\}_{k=1}^{\infty},\ t^k \in T$, and $\{\theta_k\}_{k=1}^{\infty},\ \lim\limits_{k\to\infty} \theta_k = +0$ such that

$$t^k = t + \theta_k l + \frac{1}{2}\theta_k^2 w + o(\theta_k^2) \in T. \tag{36}$$

From condition $t^k \in T$ and the convexity of $f(x,t)$ w.r.t. x in $\tilde{X} \subset \mathbb{R}^n$ for any $t \in T$, it follows that (34) holds for all $x^1, x^2 \in \tilde{X}$, and $x(\lambda) = \lambda x^1 + (1-\lambda)x^2$ with any $\lambda \in [0,1]$.

Taking into account equalities (35) that hold true for the given l and $\forall x \in \tilde{X}$, and presentation (36), we obtain from (34)

$$\begin{aligned} \tfrac{1}{2}\left(\tfrac{\partial f^T(x(\lambda),t)}{\partial t}w + l\tfrac{\partial^2 f(x(\lambda),t)}{\partial t^2}l\right)\theta_k^2 + o(\theta_k^2) \leq \\ \tfrac{1}{2}\lambda\left(\tfrac{\partial f^T(x^1,t)}{\partial t}w + l\tfrac{\partial^2 f(x^1,t)}{\partial t^2}l\right)\theta_k^2 + \\ \tfrac{1}{2}(1-\lambda)\left(\tfrac{\partial f^T(x^2,t)}{\partial t}w + l\tfrac{\partial^2 f(x^2,t)}{\partial t^2}l\right)\theta_k^2 + o(\theta_k^2). \end{aligned} \tag{37}$$

Notice that due to relations (35), problem $(LP(x,t,l))$ has an optimal solution for all $x \in \tilde{X}$ and all $l \in L_1(t)$.

Let $w(x(\lambda),l)$ be an optimal solution of the problem $(LP(x(\lambda),t,l))$. Then $w(x(\lambda),l) \in L_2(t,l)$.

Suppose in (37) that $w = w(x(\lambda),l)$, divide both parts of this inequality by $\frac{1}{2}\theta_k^2$ and pass to the limit with $\theta_k \to +0$. As the result we get

$$\begin{aligned} \tfrac{\partial f^T(x(\lambda),t)}{\partial t}w(x(\lambda),l) + l\tfrac{\partial^2 f(x(\lambda),t)}{\partial t^2}l \leq \\ \lambda(\tfrac{\partial f^T(x^1,t)}{\partial t}w(x(\lambda),l) + l\tfrac{\partial^2 f(x^1,t)}{\partial t^2}l) + \\ (1-\lambda)(\tfrac{\partial f^T(x^2,t)}{\partial t}w(x(\lambda),l) + l\tfrac{\partial^2 f(x^2,t)}{\partial t^2}l) \leq \\ \lambda(\tfrac{\partial f^T(x^1,t)}{\partial t}w(x^1,l) + l\tfrac{\partial^2 f(x^1,t)}{\partial t^2}l) + \\ (1-\lambda)(\tfrac{\partial f^T(x^2,t)}{\partial t}w(x^2,l) + \bar{l}\tfrac{\partial^2 f(x^2,t)}{\partial t^2}l). \end{aligned} \tag{38}$$

Here we took into account that $\frac{\partial f^T(x^i,t)}{\partial t}w(x(\lambda),l) \leq \frac{\partial f^T(x^i,t)}{\partial t}w(x^i,l)$, $i = 1,2$, where $w(x,l)$ is an optimal solution of problem $(LP(x,t,l))$. Taking into account notations (14) and the equality $val(LP(x,t,l)) = \frac{\partial f^T(x,t)}{\partial t}w(x,l)$, relations (38) take the form

$$F_2(x(\lambda),t,l) \leq \lambda F_2(x^1,t,l) + (1-\lambda)F_2(x^2,t,l),$$

for all $x^1, x^2 \in \tilde{X}$, and $x(\lambda) = x^1\lambda + (1-\lambda)x^2$, $\forall \lambda \in [0,1]$, $\forall l \in L(t)$. Therefore we conclude that the function $F_2(x,t,l)$ with $l \in L(t)$ is convex w.r.t. $x \in \tilde{X}$. The lemma is proved. □

Corollary 1 *Consider the convex problem (SIP). Suppose that an immobile index $t \in T^*$ satisfies condition (MFCQ). Then the functions $F_1(x,t,l)$ with $l \in L_1(t)$ and the functions $F_2(x,t,l)$ with $l \in C_0(t)$ are convex w.r.t. x in X.*

Proof. For the immobile index $t \in T^*$, by construction, conditions (35) are satisfied with $\tilde{X} = X$ and $L(t) = C_0(t)$. Hence the conclusions of Lemmas 1 and 2 hold true for $t \in T^*$, $\tilde{X} = X$, $L(t) = C_0(t)$. □

Proposition 2 *Suppose that (MFCQ) holds at $\bar{t} \in T^*$ and $q(\bar{t}, l) \leq 1, \forall l \in L_1(\bar{t}), l \neq 0$. Then there exists a vector $\bar{x} = x(\bar{t}) \in X$ such that*

$$F_1(\bar{x}, \bar{t}, l) < 0, \ \forall l \in L_1(\bar{t}) \setminus C_0(\bar{t}), \ \|l\| = 1; \tag{39}$$
$$F_2(\bar{x}, \bar{t}, l) < 0, \ \forall l \in C_0(\bar{t}), \ \|l\| = 1. \tag{40}$$

Proof. Let us first prove that there exists $\tilde{x} \in X$ such that

$$F_2(\tilde{x}, \bar{t}, l) < 0, \ \forall l \in C_0(\bar{t}), \ \|l\| = 1. \tag{41}$$

Consider the following SIP problem:

$$(SIP_*): \qquad \min_{x \in X, \mu \in \mathbb{R}} \ \mu$$
$$\text{s.t.} \quad F_2(x, \bar{t}, l) \leq \mu, \ \forall l \in C_0(\bar{t}), \ \|l\| = 1, \ \ -1 \leq \mu.$$

Notice that here X is a convex set and the function $F_2(x, \bar{t}, l) - \mu$ is convex w.r.t. $(x, \mu) \in X \times \mathbb{R}$ for all $l \in C_0(\bar{t})$ according to Corollary 1. Moreover, it is evident that there exists a vector $(\bar{x}, \bar{\mu})$, $\bar{x} \in X$, $\bar{\mu} > -1$, such that $\bar{F}_2(\bar{x}, l) - \bar{\mu} < 0$ for all $l \in C_0(\bar{t})$. Therefore, according to Theorem 4.1 from [2], there exists a set of indices

$$\{l^i, i = 1, 2, \dots, n+1\}: \ l^i \in C_0(\bar{t}), \ \|l^i\| = 1, i = 1, 2, \dots, n+1, \tag{42}$$

such that $val(SIP_*) = val(SIPD_*)$, where the discretized problem $(SIPD_*)$ has the form

$$(SIPD_*): \qquad \min_{x \in X, \ \mu \in \mathbb{R}} \ \mu$$
$$\text{s.t.} \quad F_2(x, \bar{t}, l^i) \leq \mu, \ i = 1, 2, \dots, n+1; \ \ -1 \leq \mu.$$

It follows from the assumption $q(\bar{t}, l) \leq 1, \forall l \in L_1(\bar{t}), l \neq 0$ and the definition of the set $C_0(\bar{t})$, that $q(\bar{t}, l) = 1$ for all $l \in C_0(\bar{t}), \|l\| = 1$. Then for any l^i $(i = 1, 2, \dots, n+1)$ from (42), there exists a vector $x^{(i)} \in X$ such that $F_2(x^{(i)}, \bar{t}, l^i) < 0$, $(i = 1, 2, \dots, n+1)$. Notice that relations (20b) with $t = \bar{t}$ imply the inequalities $F_2(x^{(j)}, \bar{t}, l^i) \leq 0, \ \forall j = 1, 2, \dots, n+1, \ i = 1, 2, \dots, n+1$. Consider the vector $\hat{x} = \frac{1}{n+1} \sum_{i=1}^{n+1} x^{(i)}$. Taking into account the convexity of the set X, the convexity of functions $F_2(x, \bar{t}, l^i), \ x \in X, i = 1, 2, \dots, n+1$, and the inequalities mentioned above, we get $\hat{x} \in X$, $F_2(\hat{x}, \bar{t}, l^i) \leq \frac{1}{n+1} \sum_{j=1}^{n+1} F_2(x^{(j)}, \bar{t}, l^i) < 0$, $i = 1, 2, \dots, n+1$.

Consequently, for the problem $(SIPD_*)$ there exists a feasible solution $(\hat{x}, \hat{\mu})$ such that

$$\hat{\mu} := \max\{F_2(\hat{x}, \bar{t}, l^i), \ i = 1, 2, \dots, n+1, \ -1\} < 0$$

and hence $val(SIPD_*) = val(SIP_*) < 0$. This implies the existence of $\tilde{x} \in X$ satisfying (41).

Now let us prove that there exists a vector $\tilde{\tilde{x}} \in X$ such that

$$F_1(\tilde{\tilde{x}}, \bar{t}, l) < 0, \forall l \in L_1(\bar{t}) \setminus C_0(\bar{t}), \|l\| = 1. \tag{43}$$

Notice that

$$\begin{gathered}\{l \in L_1(\bar{t}) \setminus C_0(\bar{t}),\ \|l\| = 1\} = \\ \{l \in \mathbb{R}^p : l = \sum_{i \in P(\bar{t})} \beta_j b_i(\bar{t}) + \sum_{i \in I(\bar{t})} \alpha_i a_i(\bar{t}), \alpha_i \geq 0, i \in I(\bar{t}), \sum_{i \in I_*(\bar{t})} \alpha_i > 0, \|l\| = 1\}.\end{gathered} \tag{44}$$

Since by construction, $q(\bar{t}, a_i(\bar{t})) = 0,\ i \in I_*(\bar{t})$, then for any $a_i(\bar{t}), i \in I_*(\bar{t})$, there exists $\bar{x}^{(i)} \in X$ such that

$$F_1(\bar{x}^{(i)}, \bar{t}, a_i(\bar{t})) < 0,\ i \in I_*(\bar{t}). \tag{45}$$

Consider vector $\tilde{\tilde{x}} = \frac{1}{|I_*(\bar{t})|} \sum\limits_{i \in I_*(\bar{t})} \bar{x}^{(i)}$. By the convexity of X, it holds: $\tilde{\tilde{x}} \in X$. Since for all $l \in L_1(\bar{t})$, function $F_1(x, \bar{t}, l)$ is convex w.r.t. $x \in X$, we have

$$\begin{gathered}F_1(\tilde{\tilde{x}}, \bar{t}, l) \leq \frac{1}{|I_*(\bar{t})|} \sum_{s \in I_*(\bar{t})} F_1(\bar{x}^{(s)}, \bar{t}, l) = \\ \frac{1}{|I_*(\bar{t})|} \sum_{s \in I_*(\bar{t})} (\sum_{i \in I_*(\bar{t})} F_1(\bar{x}^{(s)}, \bar{t}, a_i(\bar{t}))\alpha_i).\end{gathered} \tag{46}$$

Here we took into account representation (16), the linearity of the function $F_1(x, \bar{t}, l)$ w.r.t. l, and the equalities $F_1(x, \bar{t}, b_i(\bar{t})) = 0, i \in P(\bar{t}),\ F_1(x, \bar{t}, a_i(\bar{t})) = 0, i = I_0(\bar{t}), \forall x \in X$, that should be satisfied by construction. It is easy to confirm that (20a) (with $t = \bar{t}$) and (44)-(46) imply (43).

Now let us consider vector $\bar{x} = (\tilde{x} + \tilde{\tilde{x}})/2$. It is evident that $\bar{x} \in X$. From the convexity of the function $F_1(x, \bar{t}, l)$ w.r.t. $x \in X$ for all $l \in L_1(\bar{t})$, and from relations (43), (20a) with $t = \bar{t}$, it follows that inequalities (39) hold true.

From the convexity of the function $F_2(x, \bar{t}, l)$ w.r.t. $x \in X$ for all $l \in C_0(\bar{t})$ and from relations (41), (20b) with $t = \bar{t}$ it follows that inequalities (40) hold true. The proposition is proved. □

Let us make an additional assumption on the immobility orders of the immobile indices of problem (SIP) (see Definition 3).

Assumption 2 *Given problem (SIP), for all $t \in T^*$, it holds $q(t, l) \leq 1$, $\forall l \in L_1(t)$, $l \neq 0$.*

We consider that the assumption is fulfilled if $T^* = \emptyset$.

Corollary 2 *Suppose that the convex problem (SIP) satisfies Assumptions 1 and 2. Then the set of immobile indices T^* either is empty or contains a finite number of indices.*

Proof. Notice that under Assumptions 1 and 2, from Proposition 2, it follows that for any $\bar{t} \in T^*$, there exists $\bar{x} = x(\bar{t}) \in X$ such that the vector $\bar{t}$ is a strict local maximizer in the lower level problem $(LLP(\bar{x}))$, i.e. there exists $\delta_0 > 0$ such that

$$f(\bar{x}, \bar{t}) > f(\bar{x}, t), \ \forall t \in T \setminus \{\bar{t}\}, \ \|t - \bar{t}\| \leq \delta_0. \tag{47}$$

Suppose that $|T^*| = \infty$. Since T is compact, then there exists a convergent sequence of indices $t^k \in T^*, \ k = 1, 2, \dots$ such that $\bar{t} := \lim\limits_{k\to\infty} t^k, \ t^k \neq \bar{t}, \quad \bar{t} \in T$. The function $f(x,t)$ is continuous w.r.t t for all $x \in X$ and hence $\bar{t} \in T^*$. Then we can conclude that for a sufficiently large k, it holds $0 = f(x, \bar{t}) = f(x, t^k), \ \|t^k - \bar{t}\| \leq \delta_0$ for all $x \in X$. But the last relation contradicts condition (47) that should be satisfied for any $\bar{t} \in T^*$ and $t^k : \|t^k - \bar{t}\| \leq \delta_0$, and hence the corollary is proved. $\square$

4.2 A Slater type condition

Suppose that (the convex) problem (SIP) satisfies the Assumptions 1 and 2. Then from Corollary 2, it follows that the set T^* of immobile indices in problem (SIP) admits a presentation

$$T^* = \{t^*_j, j \in J_*\}, \quad \text{where } |J_*| < \infty. \tag{48}$$

For the sake of simplicity, let us introduce the following notations:

$$F_{1j}(x,l) := F_1(x, t^*_j, l), \ F_{2j}(x,l) := F_2(x, t^*_j, l), \ j \in J_*. \tag{49}$$

Lemma 3 *Suppose that the Assumptions 1 and 2 are satisfied. Then there exists a vector $\tilde{x} \in X$ such that*

$$F_{1j}(\tilde{x}, l) < 0, \ \forall l \in L_1(t^*_j) \setminus C_0(t^*_j), \ \|l\| = 1; \tag{50}$$

$$F_{2j}(\tilde{x}, l) < 0, \ \forall l \in C_0(t^*_j), \ \|l\| = 1, \ j \in J_*; \tag{51}$$

$$f(\tilde{x}, t) < 0, \ t \in T \setminus T^*. \tag{52}$$

Proof. First, suppose that $T^* \neq \emptyset$. From Proposition 2, it follows that for each $j \in J_*$ there exists $x^{(j)} \in X$ such that

$$\begin{gathered} F_{1j}(x^{(j)}, l) < 0, \ \forall l \in L_1(t^*_j) \setminus C_0(t^*_j), \ \|l\| = 1; \\ F_{2j}(x^{(j)}, l) < 0, \ \forall l \in C_0(t^*_j), \ \|l\| = 1. \end{gathered} \tag{53}$$

Notice that from the optimality conditions ($15a$) and ($20b$), it follows that for any $x \in X$ it holds

$$F_{1j}(x,l) \leq 0, \ \forall l \in L_1(t^*_j), \ \ F_{2j}(x,l) \leq 0, \ \forall l \in C_0(t^*_j), \ \ j \in J_*. \tag{54}$$

Consider the vector $\breve{x} = \sum\limits_{j \in J_*} \frac{x^{(j)}}{|J_*|}$. It is evident that $\breve{x} \in X$. Given $j \in J_*$, from convexity w.r.t. $x \in X$ of the functions $F_{1j}(x,l)$ with any $l \in L_1(t^*_j)$ and $F_{2j}(x,l)$, with any $l \in L^0(t^*_j)$, and relations (53),(54), we get

$$\begin{gathered} F_{1j}(\breve{x},l) < 0,\ \forall l \in L_1(t^*_j) \setminus C_0(t^*_j),\ F_{1j}(\breve{x},l) = 0,\ \forall l \in C_0(t^*_j),\ \|l\| = 1; \\ F_{2j}(\breve{x},l) < 0,\ \forall l \in C_0(t^*_j),\ \|l\| = 1,\ j \in J_*. \end{gathered} \tag{55}$$

From (55), it follows that the immobile indices $t^*_j, j \in J_*$, are the strict local maximizers in problem $(LLP(\breve{x}))$, i.e., there exists $\varepsilon > 0$ such that

$$0 = f(\breve{x}, t^*_j) > f(\breve{x}, t), \forall t \in T_\varepsilon(j) \setminus t^*_j,\ j \in J_*; \quad f(\breve{x}, t) \le 0,\ t \in T(\varepsilon). \tag{56}$$

Here

$$T_\varepsilon(j) = \{t \in T:\ \|t - t^*_j\| \le \varepsilon\},\ j \in J_*,\ T(\varepsilon) = T \setminus \bigcup_{j \in J_*} \text{int } T_\varepsilon(j). \tag{57}$$

Consider problem

$$(SIP_{**}): \quad \min_{x \in X} \mu \quad \text{s.t.} \quad f(x,t) \le \mu, \forall t \in T(\varepsilon);\ -1 \le \mu.$$

Notice that X is a convex set and the function $f(x,t) - \mu$ is convex w.r.t. (x, μ). It is evident that there exists a vector $(\bar{x}, \bar{\mu})$, $\bar{x} \in X$, $\bar{\mu} > -1$, such that $f(\bar{x}, t) - \bar{\mu} < 0,\ t \in T(\varepsilon)$. Therefore the problem (SIP_{**}) verifies the conditions of Theorem 4.1 from [2]. Then, there exists a set of indices $t^{(i)} \in T(\varepsilon),\ i = 1, 2, \ldots, n+1$, such that $val(SIP_{**}) = val(SIPD_{**})$, where problem $(SIPD_{**})$ has the form

$$(SIPD_{**}): \quad \min_{x \in X} \mu \quad \text{s.t.} \quad f(x, t^{(i)}) \le \mu,\ i = 1, 2, \ldots, n+1;\ -1 \le \mu.$$

By construction, $t^{(i)} \notin T^*,\ i = 1, 2, \ldots, n+1$, hence for each index $t^{(i)}$ there exists $x^{(i)} \in X$ such that

$$f(x^{(i)}, t^{(i)}) < 0,\ i = 1, 2, \ldots, n+1.$$

Recall that $f(x, t^{(i)}) \le 0, i = 1, 2, \ldots, n+1,\ \forall x \in X$. Consider vector $x^* = \sum\limits_{i=1}^{n+1} \frac{x^{(i)}}{n+1}$. It is evident that $x^* \in X$. Then taking into account the inequalities above and the convexity of the functions $f(x, t^{(i)}),\ i = 1, 2, \ldots, n+1$ w.r.t. $x \in X$, we get $f(x^*, t^{(i)}) < 0,\ i = 1, 2, \ldots, n+1$.Consequently, the problem $(SIPD_{**})$ has a feasible solution (x^*, μ^*) where $\mu^* = \max\{-1,\ f(x^*, t^{(i)}),\ i = 1, 2, \ldots, n+1\} < 0$. Hence, $val(SIPD_{**}) = val(SIP_{**}) < 0$ and there exists a vector $\hat{x} \in X$, such that

$$f(\hat{x}, t) < 0,\ t \in T(\varepsilon), \quad f(\hat{x}, t) \le 0,\ t \in T_\varepsilon(j),\ j \in J_*. \tag{58}$$

Consider vector $\tilde{x} = (\hat{x} + \breve{x})/2 \in X$. Taking into account the convexity w.r.t. x of the function $f(x,t), x \in X$, for all $t \in T$ and formulas (56), (58), we conclude that (52) holds.

Finally, the convexity of functions $F_{1j}(x,l),\ F_{2j}(x,l),\ j \in J_*$, w.r.t. $x \in X$ for the corresponding vectors l, and relations (54), (55) imply inequalities (50), (51).

Now suppose that $T^* = \emptyset$ or, equivalently, $J_* = \emptyset$. Then in (57) we have $T(\varepsilon) = T$ and relations (58) take the form $f(\hat{x},t) < 0,\ t \in T$. Hence inequalities (52) are satisfied with $\tilde{x} = \hat{x}$. □

Corollary 3 *Given the convex problem* (SIP)*, suppose that* $T^* = \emptyset$*. Then the constraints of this problem satisfy the Slater condition, i.e.*

$$\exists\, \hat{x} \in \mathbb{R}^n \text{ such that } f(\hat{x},t) < 0,\ t \in T. \tag{59}$$

It is evident that in the case $T^* \neq \emptyset$ the constraints of problem (SIP) do not satisfy the Slater condition (59) since $f(x,t) = 0$ for all $x \in X$ and all $t \in T^*$. However, Lemma 3 shows that is this case under the Assumptions 1 and 2 there exists $\tilde{x} \in X$ satisfying conditions (50)-(52). We will refer to these conditions as a *Slater type condition.*

4.3 Convexity of the set defined by the equality constraints of the auxiliary NLP problem (23)

Finally, we will establish one more important property of the auxiliary NLP problem (23).

Suppose that the set T^* of immobile indices in the problem (SIP) is finite, i.e. it admits a presentation (48). For sake of simplicity, in what follows, for each $j \in J_*$, we will use notation

$$b_i(j) := b_i(t_j^*), i \in P(j) := P(t_j^*),\ a_i(j) := a_i(t_j^*), i \in I(j) := I(t_j^*),$$
$$I_0(j) := I_0(t_j^*),\ I_*(j) := I_*(t_j^*),$$

where $b_i(t),\ i \in P(t),\ |P(t)| < \infty$, are the bidirectional extremal rays and $a_i(t), i \in I(t),\ |I(t)| < \infty$, are the unidirectional extremal rays of the cone $L_1(t)$ with $t \in T$, and the sets $I_0(t),\ I_*(t)$ are defined in (17).

Lemma 4 *Suppose that* $T^* = \{t_j^*, j \in J_*\},\ |J_*| < \infty$*, and the Assumption 1 is fulfiled. Then there exist subsets*

$$\bar{J} \subset J_*,\ \bar{I}(j) \subset I_0(j), j \in \bar{J}, \tag{60}$$

such that the set

$$\begin{aligned} X^* = \Big\{ & x \in \mathbb{R}^n : \ f(x,t_j^*) \le 0, j \in J_* \backslash \bar{J}; f(x,t_j^*) = 0, \\ & \frac{\partial f^T(x,t_j^*)}{\partial t} b_i(j) = 0, i \in P(j), \frac{\partial f^T(x,t_j^*)}{\partial t} a_i(j) = 0, i \in \bar{I}(j), \\ & \frac{\partial f^T(x,t_j^*)}{\partial t} a_i(j) \le 0, i \in I_0(j) \backslash \bar{I}(j), j \in \bar{J} \Big\} \end{aligned} \tag{61}$$

is convex and for $M := |J_* \setminus \bar{J}| + \sum_{j\in\bar{J}} |I_0(j) \setminus \bar{I}(j)|$ *one of the following conditions is fulfilled:*

A) $M = 0$, *or* **B)** $M \geq 1$ *and there exists a vector* $z \in X^*$ *such that*

$$f(z,t_j^*) < 0, j \in J_* \setminus \bar{J}, \ \frac{\partial f^T(z,t_j^*)}{\partial t} a_i(j) < 0, i \in I_0(j)\setminus\bar{I}(j), j \in \bar{J}. \tag{62}$$

Proof. Suppose that indices t_j^* and vectors $b_i(j), i \in P(j),\ a_i(j), i \in I_0(j), j \in J_*$ are given.

If $J_* = \emptyset$ then the statement of the Lemma is trivially satisfied since $X^* = \mathbb{R}^n$ and condition **A)** is fulfilled.

Suppose that $J_* \neq \emptyset$. We will prove the statement algorithmically. At the beginning, set $J^{(0)} = \emptyset,\ k = 0$ and consider the set

$$X^{(1)} = \{x \in \mathbb{R}^n : \ f(x,t_j^*) \leq 0, j \in J_*\}.$$

It is evident that $X^{(1)}$ is convex.

At the (k+1)-th iteration, we have the index sets $J^{(k)} \subset J_*,\ I^{(k)}(j) \subset I_0(j), j \in J^{(k)}$, and the set

$$\begin{aligned} X^{(k+1)} =\Big\{ & x \in \mathbb{R}^n : \ f(x,t_j^*) \leq 0, j \in J_*\setminus J^{(k)}; f(x,t_j^*) = 0, \\ & \frac{\partial f^T(x,t_j^*)}{\partial t} b_i(j) = 0, i \in P(j), \frac{\partial f^T(x,t_j^*)}{\partial t} a_i(j) = 0, i \in I^{(k)}(j), \\ & \frac{\partial f^T(x,t_j^*)}{\partial t} a_i(j) \leq 0, i \in I_0(j)\setminus I^{(k)}(j), j \in J^{(k)}\Big\}. \end{aligned} \tag{63}$$

Notice that for $k = 0$ we have $J^{(k)} = \emptyset$ and hence we do not construct the sets $I^{(k)}(j), j \in J^{(k)}$.

Suppose that $X^{(k+1)}$ is convex. Set

$$\Delta J^{(k+1)} = \{j \in J_*\setminus J^{(k)} : \ f(x,t_j^*) = 0, \forall x \in X^{(k+1)}\}, \tag{64}$$

$$\Delta I^{(k+1)}(j) = \{i \in I_0(j)\setminus I^{(k)}(j) : \ \tfrac{\partial f^T(x,t_j^*)}{\partial t} a_i(j) = 0, \forall x \in X^{(k+1)}\}, j \in J^{(k)}.$$

In the case $|\Delta J^{(k+1)}| + \sum\limits_{j\in J^{(k)}} |\Delta I^{(k+1)}(j)| > 0$ we set

$$J^{(k+1)} = J^{(k)} \bigcup \Delta J^{(k+1)},\ I^{(k+1)}(j) = \emptyset, j \in \Delta J^{(k+1)},$$

$$I^{(k+1)}(j) = I^{(k)}(j) \bigcup \Delta I^{(k+1)}(j), j \in J^{(k)}.$$

It follows from (64) that the set $X^{(k+1)}$ defined in (63), can be rewritten in the form

$$\begin{aligned} X^{(k+1)} = \{ & x \in \mathbb{R}^n : \ f(x,t_j^*) \leq 0, j \in J_* \setminus J^{(k+1)}; \ f(x,t_j^*) = 0, j \in J^{(k+1)}; \\ & \frac{\partial f^T(x,t_j^*)}{\partial t} b_i(j) = 0, i \in P(j), \frac{\partial f^T(x,t_j^*)}{\partial t} a_i(j) = 0, i \in I^{(k+1)}(j), \\ & \frac{\partial f^T(x,t_j^*)}{\partial t} a_i(j) \leq 0, i \in I_0(j)\setminus I^{(k+1)}(j), j \in J^{(k)}\}. \end{aligned}$$

By assumption, this set is convex and by construction, we have $f(x,t^*_j)=0, \forall x\in X^{(k+1)}, j\in \Delta J^{(k+1)}$. From Lemma 1, it follows that for $j\in\Delta J^{(k+1)}$, the functions $\frac{\partial f^T(x,t^*_j)}{\partial t}l, x\in X^{(k+1)}$, are convex w.r.t. $x\in X^{(k+1)}$ for all $l\in L_1(t^*_j)$, i.e. for all $x^1,x^2\in X^{(k+1)}$ and $x(\lambda)=\lambda x^1+(1-\lambda)x^2,\ \forall\lambda\in[0,1]$, it holds

$$\frac{\partial f^T(x(\lambda),t^*_j)}{\partial t}l\leq\lambda\frac{\partial f^T(x^1,t^*_j)}{\partial t}l+(1-\lambda)\frac{\partial f^T(x^2,t^*_j)}{\partial t}l,\ j\in\Delta J^{(k+1)}. \quad (65)$$

By construction, $\pm b_i(j)\in L_1(t^*_j), i\in P(j),\ a_i(j)\in L_1(t^*_j),\ i\in I_0(j),\ j\in\Delta J^{(k+1)}$. Having supposed $l=\pm b_i(j),\ i\in P(j)$, and $l=a_i(j), i\in I_0(j)$, in (65), we get for $j\in\Delta J^{(k+1)}$,

$$\begin{aligned}&\tfrac{\partial f^T(x(\lambda),t^*_j)}{\partial t}b_i(j)=\lambda\tfrac{\partial f^T(x^1,t^*_j)}{\partial t}b_i(j)+(1-\lambda)\tfrac{\partial f^T(x^2,t^*_j)}{\partial t}b_i(j),\ i\in P(j),\\ &\tfrac{\partial f^T(x(\lambda),t^*_j)}{\partial t}a_i(j)\leq\lambda\tfrac{\partial f^T(x^1,t^*_j)}{\partial t}a_i(j)+(1-\lambda)\tfrac{\partial f^T(x^2,t^*_j)}{\partial t}a_i(j), i\in I_0(j).\end{aligned} \quad (66)$$

Let us construct the new set

$$\begin{aligned}X^{(k+2)}:=\big\{x\in\mathbb{R}^n:{}&f(x,t^*_j)\leq 0, j\in J_*\setminus J^{(k+1)};\\ &f(x,t^*_j)=0,\frac{\partial f^T(x,t^*_j)}{\partial t}b_i(j)=0, i\in P(j),\\ &\frac{\partial f^T(x,t^*_j)}{\partial t}a_i(j)=0, i\in I^{(k+1)}(j),\\ &\frac{\partial f^T(x,t^*_j)}{\partial t}a_i(j)\leq 0, i\in I_0(j)\setminus I^{(k+1)}, j\in J^{(k+1)}\big\}= \quad (67)\\ &X^{(k+1)}\bigcap\big\{x\in\mathbb{R}^n:\tfrac{\partial f^T(x,t^*_j)}{\partial t}b_i(j)=0, i\in P(j),\\ &\tfrac{\partial f^T(x,t^*_j)}{\partial t}a_i(j)\leq 0, i\in I_0(j), j\in\Delta J^{(k+1)}\big\}.\end{aligned}$$

From (66), (67), and the convexity of the set $X^{(k+1)}$, it follows that the set $X^{(k+2)}$ is convex.

Set $k:=k+1$ and pass to the next iteration of the algorithm.

In the case $|\Delta J^{(k+1)}|+\sum\limits_{j\in J^{(k)}}|\Delta I^{(k+1)}(j)|=0$ we stop at the current iteration and set

$$X^*=X^{(k+1)}, \bar J=J^{(k+1)},\ \bar I(j)=I^{(k+1)}(j), j\in\bar J.$$

By construction, the set X^* is convex. If the condition **A)** is fulfilled, then the Lemma is proved.

Suppose that the condition **A)** is not satisfied. Let us show that the condition **B)** is fulfilled. By construction, for each $j\in J_*\setminus\bar J$ there exists a vector $z^j\in X^*$ such that $f(z^j,t^*_j)<0$ and for each $i\in I_0(j)\setminus\bar I(j),\ j\in\bar J$, there exists a vector $z^{ij}\in X^*$ such that $\frac{\partial f^T(z^{ij},t^*_j)}{\partial t}a_i(j)<0$. Taking into account these inequalities, convexity of the set X^*, and convexity of functions

$f(x,t_j^*), j \in J_* \setminus \bar{J}$, $\frac{\partial f^T(x,t_j^*)}{\partial t}a_i(j)$, $i \in I_0(j)\setminus \bar{I}(j)$, $j \in \bar{J}$, we conclude that relations (62) are satisfied with $z = (\sum_{j\in J_*\setminus \bar{J}} z^j + \sum_{j\in \bar{J}}\sum_{i\in I_0(j)\setminus \bar{I}(j)} z^{ij})/M \in X^*$. $\Box$

Suppose that the Assumptions 1 and 2 are satisfied. Then from Corollary 2, it follows that the set T^* of immobile indices in the problem (SIP) is finite and admits representation (48). Hence without loss of generality we can set $\bar{T} = T^*$ in Theorem 2.

Let us consider the set Q defined by the equality constraints of the NLP problem (23) with $\bar{T} = T^*$:

$$\begin{aligned} Q = Q(T^*) = &\{x \in \mathbb{R}^n : f(x,t_j^*) = 0,\ \frac{\partial f^T(x,t_j^*)}{\partial t}b_i(j) = 0, i \in P(j), \\ &\frac{\partial f^T(x,t_j^*)}{\partial t}a_i(j) = 0, i \in I_0(j), j \in J_*\}. \end{aligned} \tag{68}$$

Lemma 5 *Suppose that the Assumptions 1 and 2 are fulfilled. Then the set Q defined in (68) is convex.*

Proof. It follows from Lemma 4 that there exist sets (60) such that the set X^* (62) is convex and the condition **A)** or **B)** is fulfilled.

It is easy to show that under Assumption 2 (that is stronger than the assumption $|T^*| < \infty$) the condition **A)** is fulfilled, since otherwise we get the contradiction with the fact that the indices $t_j^*, j \in J_*\setminus\bar{J}$, are immobile and with the condition $q(t_j^*, a_i(j)) > 0,\ i \in I_0(j)\setminus \bar{I}(j),\ j \in \bar{J}$. It is evident that under condition **A)**, the set X^* coincides with Q. By Lemma 4 the set X^* is convex, consequently the set Q is convex as well. $\Box$

Corollary 4 *Suppose that the Assumptions 1 and 2 are satisfied. Then for all $j \in J_*$, the auxiliary functions $F_1(x,t_j^*,l)$ with $l \in L_1(t_j^*)$, are convex w.r.t. x in Q and the functions $F_2(x,t_j^*,l)$ with $l \in C_0(t_j^*)$, are convex w.r.t. x in $\bar{Q}$ with*

$$\bar{Q} = \{x \in Q : \frac{\partial f^T(x,t_j^*)}{\partial t}a_i(j) \le 0, i \in I_*(j), j \in J_*\}.$$

Proof. It was proved that the set Q is convex and by construction, for every $j \in J_*$, the equalities $f(x,t_j^*) = 0\ \ \forall x \in Q$ hold true. Hence, by Lemma 1, the auxiliary functions $F_1(x,t_j^*,l)$ with $l \in L_1(t_j^*)$, are convex w.r.t. x in Q for every $j \in J_*$. Moreover, taking into account the convexity of these functions, we conclude that the set $\bar{Q}$ is convex and by construction, for every $j \in J_*$, condition (35) is satisfied for $t = t_j^* \in T^*$, $\tilde{X} = \bar{Q},\ L(t) = C_0(t_j^*)$. Hence, from Lemma 2, we conclude that for every $j \in J_*$, the auxiliary functions $F_2(x,t_j^*,l)$ with $l \in C_0(t_j^*)$, are convex w.r.t. x in $\bar{Q}$. $\Box$

5 Conclusions and the future work

In this paper we studied the convex smooth problem (SIP) with the finitely representable compact index set. In section 3.1, under Assumption 1 we have formulated sufficient optimality conditions for this problem in terms of the optimality conditions for the auxiliary NLP problem (23) (Theorem 2).

If, additionally, Assumption 2 holds, then we can reformulate problem (23) with $\bar{T} = T^* = \{t_j^*, j \in J_*\}$ as follows:

$$\min_{x \in Q} c(x) \tag{69}$$
$$\text{s.t. } F_{1j}(x, a_i(j)) \le 0, i \in I_*(j),\ F_{2j}(x, l_k(j)) \le 0, k = 1, ..., m(t_j^*),\ j \in J_*,$$
$$f(x,t) \le 0, t \in T^0,$$

where the constraint functions are defined in (14) and (49), vectors $l_k(j) = l_k(t_j^*), k = 1, ..., m(t_j^*)$, $j \in J_*$, are defined in (21),(22), and the set Q is defined in (68).

In Section 4, it was established that the NLP problem (69) possesses the following properties:

• the set Q is convex,

• the inequality constraint functions $F_{1j}(x, a_i(j)), i \in I_*(j),\ j \in J_*$, and $f(x,t), t \in T^0$, are convex w.r.t. x in Q, and the functions $F_{2j}(x, l_k(j))$, $k = 1, ..., m(t_j^*), j \in J_*$, are convex w.r.t. x in $\bar{Q} = \{x \in Q : F_{1j}(x, a_i(j)) \le 0, i \in I_*(j),\ j \in J_*\}$,

• there exists $\tilde{x} \in X \subset Q$ such that

$$F_{1j}(\tilde{x}, a_i(j)) < 0, i \in I_*(j);\ F_{2j}(\tilde{x}, l_k(j)) < 0, k = 1, ..., m(t_j^*),\ j \in J_*; \tag{70}$$
$$f(\tilde{x}, t) < 0, t \in T^0.$$

It follows from these properties that problem (69) is a convex programming problem and its constraints satisfy the Slater's type condition (70). Notice that taking into account the convexity of problem (69), it is easy to show (see Theorem 5.100 from [1]) that without loss of generality, condition (22) can be replaced by (24).

Using the properties of problem (69), one can formulate and prove efficient optimality conditions for its feasible solution $x^0 \in X$, that according to Theorem 2 will provide new sufficient optimality conditions for $x^0 \in X$ in the original SIP problem (SIP). Such new conditions may be of special interest since they may be proved without additional CQs. We are going to do it in our subsequent paper.

Finally, we would like to make some remarks.

– The regularity condition ($MFCQ$) for the lower level problem ($LLP(x)$) in Assumption 1 can be replaced by another even less restrictive CQs.
– We have not used Assumption 2 to prove the sufficient optimality conditions for the problem (SIP). This assumption was introduced only to guarantee a finiteness of the set of immobile indices and to prove the existence of $\tilde{x} \in X$ satisfying inequalities (70).
– Theorem 4 that is proved in the Appendix and the example presented in section 3, show that, for convex SIP problems, the sufficient optimality conditions proved in the paper are more efficient in comparison with the

conditions from [11] (some of the strongest sufficient optimality conditions known from the literature) since the fulfillment of the conditions from [11] imply the fulfillment of the optimality conditions proved in the paper (Theorem 2) while the converse is not true.

Appendix

First, we reformulate the sufficient optimality conditions from the paper [11] devoted to problems of GSIP, for the problem (SIP).

Theorem 3 *(Theorem 5.1 from [11].) Let* $x^0 \in X$ *and assume that for all* $t \in T_a(x^0)$ *(LICQ) is valid:*

$$\text{(LICQ):} \quad \text{the vectors } \frac{\partial g_s(t)}{\partial t},\ s \in S_a(t), \quad \text{are linearly independent.}$$

Suppose that for every $\xi \in \mathcal{K}$, $\xi \neq 0$,

$$\mathcal{K} := \{\xi \in \mathbb{R}^n : \xi^T \frac{\partial c(x^0)}{\partial x} \leq 0, \xi^T \frac{\partial f(x^0,t)}{\partial x} \leq 0,\ t \in T_a(x^0)\},$$

there exists a set of points

$$t_j \in T_a(x^0), j \in J,\ |J| < \infty, \tag{71}$$

and vector of multipliers $\lambda = (\lambda_0 \geq 0, \lambda_j \geq 0, j \in J)$ *such that*

$$\text{(i)} \quad \frac{\partial c(x^0)}{\partial x}\lambda_0 + \sum_{j\in J} \frac{\partial f(x^0,t_j)}{\partial x}\lambda_j = 0, \tag{72}$$

$$\text{(ii)} \quad \xi^T \frac{\partial^2 \mathcal{L}(x^0,\lambda,J)}{\partial x^2}\xi + 2\sum_{j\in J}\lambda_j Q(x^0,\xi,t_j,y(j)) > 0, \tag{73}$$

where $\mathcal{L}(x,\lambda,J) = c(x)\lambda_0 + \sum\limits_{j\in J} f(x,t_j)\lambda_j$,

$$Q(x^0,\xi,t_j,y(j)) = \max\left(\frac{1}{2}\eta^T \frac{\partial^2 L(x^0,t_j,y(j))}{\partial t^2}\eta + \xi^T \frac{\partial^2 f(x^0,t_j)}{\partial x \partial t}\eta\right) \tag{74}$$

$$\begin{aligned} \text{s.t.} \quad & \eta^T \frac{\partial g_s(t_j)}{\partial t} = 0 \ \textit{if}\ y_s(j) > 0, \\ & \eta^T \frac{\partial g_s(t_j)}{\partial t} \leq 0 \ \textit{if}\ y_s(j) = 0, s \in S_a(t_j), \end{aligned}$$

and $L(x,t_j,y(j)) = f(x,t_j) - \sum\limits_{s\in S_a(t_j)} y_s(j) g_s(t_j)$, $y(j) = (y_s(j), s \in S_a(t_j))$ *is a unique vector satisfying the conditions* $\frac{\partial L(x^0,t_j,y(j))}{\partial t} = 0$, $y_s(j) \geq 0$, $y_s(j) g_s(t_j) = 0$, $j \in S_a(t_j)$.

Then x^0 *is a strict local minimum of (SIP).*

Finally, for the sake of completeness, let us prove that for the convex (SIP) problems with finitely representable index sets, the sufficient optimality conditions from Theorem 3 ensure the fulfillment of the optimality conditions proved in the paper.

Theorem 4 *Suppose that the functions $c(x)$ and $f(x,t)$ for all $t \in T$ are convex w.r.t. $x \in \mathbb{R}^n$, $|T_a(x^0)| < \infty$ and all the assumptions of Theorem 3 are fulfilled for $x^0 \in X$. Then there exist subsets*

$$\bar{J} \subset J_*,\ \bar{I}(j) \subset I_0(j), j \in \bar{J}, \tag{75}$$

and

$$\{t_j, j \in J^0\} \subset T_a(x^0) \setminus T^*,\ |J^0| \leq n, \tag{76}$$

such that x^0 is optimal in the following convex NLP problem

$$\begin{gathered}\min\ c(x),\\ \text{s.t. } f(x,t_j^*) = 0,\ \tfrac{\partial f(x,t_j^*)}{\partial t} b_i(j) = 0, i \in P(j),\ \tfrac{\partial f(x,t_j^*)}{\partial t} a_i(j) = 0, i \in \bar{I}(j),\\ \tfrac{\partial f(x,t_j^*)}{\partial t} a_i(j) \leq 0, i \in I(j) \setminus \bar{I}(j),\ j \in \bar{J};\\ f(x,t_j^*) \leq 0, j \in J_* \setminus \bar{J},\ f(x,t_j) \leq 0, j \in J^0.\end{gathered} \tag{77}$$

Here $T^* = \{t_j^*, j \in J_*\}$, $|J_*| < \infty$, is the set of immobile indices in problem (SIP).

Proof. It is evident that (LICQ) implies the fulfillment of Assumption 1. Hence assumptions of Lemma 4 are satisfied.

First let us consider that J^0 is such that $\{t_j, j \in J^0\} = T_a(x^0) \setminus T^*$ and the subsets (75) coincide with the sets (60) from Lemma 4. Using the set X^* (see (61)) let us rewrite the problem (77) is follows

$$\begin{gathered}\min\ c(x),\\ \text{s.t. } x \in X^*, \tfrac{\partial f(x,t_j^*)}{\partial t} a_i(j) \leq 0, i \in I(j) \setminus I_0(j), j \in \bar{J};\ f(x,t_j) \leq 0, j \in J^0.\end{gathered} \tag{78}$$

Notice that the set X^* is convex, the functions $\frac{\partial f(x,t_j^*)}{\partial t} a_i(j), i \in I(j) \setminus I_0(j),\ j \in \bar{J}$, and $f(x,t_j), j \in J^0$, are convex w.r.t. x in X^* and $|J^0| < \infty$ (due to assumption $|T_a(x^0)| < \infty$).

Suppose that all the assumptions of the theorem are fulfilled but the vector x^0 is not optimal in problem (78). Let x^* be a feasible solution of this problem such that $c(x^*) < c(x^0)$. Then taking into account the convexity of problem (78) it is easy to show that the vector $\xi = x^* - x^0$ satisfies the conditions

$$\xi^T \frac{\partial c(x^0)}{\partial x} < 0,\ \xi^T \frac{\partial f(x^0,t)}{\partial x} \leq 0, t \in T_a(x^0) \setminus \{t_j^*, j \in \bar{J}\}, \tag{79}$$

$$\xi^T \frac{\partial f(x^0,t_j^*)}{\partial x} = 0,\ \xi^T \frac{\partial^2 f(x^0,t_j^*)}{\partial x^2} \xi = 0, j \in \bar{J}, \tag{80}$$

$$\xi^T \frac{\partial^2 f(x^0,t_j^*)}{\partial x \partial t} b_i(j) = 0, i \in P(j);\ \xi^T \frac{\partial^2 f(x^0,t_j^*)}{\partial x \partial t} a_i(j) \leq 0, i \in I_0(j), j \in \bar{J}, \tag{81}$$

$$\xi^T \frac{\partial^2 f(x^0,t_j^*)}{\partial x \partial t} a_i(j) \leq 0 \text{ if } \frac{\partial f^T(x^0,t_j^*)}{\partial t} a_i(j) = 0, i \in I(j) \setminus I_0(j), j \in \bar{J}. \tag{82}$$

It follows from (79), (80) that $\xi \in \mathcal{K} \setminus 0$. Hence according to the assumptions of the theorem, relations (72) and (73) should be satisfied for ξ with some set (71) and some vector $\lambda = (\lambda_0, \lambda_j, j \in J) \geq 0$. For simplicity, without loss of generality, in what follows, we will suppose that the set (71) coincides with the set $T_a(x^0)$ i.e. $J = J^0 \cup J_*$ and $t_j = t_j^*, j \in J_*$.

Multiplying both sides of (72) by ξ^T, we get $\xi^T \frac{\partial c(x^0)}{\partial x}\lambda_0 + \sum\limits_{j \in J} \xi^T \frac{\partial f(x^0,t)}{\partial x}\lambda_j = 0$. From this equality and (79), (80) it follows that $\lambda_0 = 0$ and condition (72) takes the form

$$\sum_{j \in J} \frac{\partial f(x^0, t_j)}{\partial x}\lambda_j = 0. \tag{83}$$

From the statements of Lemma 4, the inclusion $X \subset X^*$ and the definition of immobile indices, it follows that for any finite set (71) there exists a vector $z^* \in X^*$ such that $f(z^*, t_j) < 0, j \in J \setminus \bar{J}$. Taking into account these inequalities and the equalities $f(x^0, t_j) = 0,\ j \in J$, convexity of the set X^* and convexity of $f(x,t)$ w.r.t. x in $\mathbb{R}^n$ for all $t \in T$, we conclude that the vector $h = z^* - x^0$ satisfies

$$h^T \frac{\partial f(x^0, t_j)}{\partial x} = 0, j \in \bar{J},\ \ h^T \frac{\partial f(x^0, t_j)}{\partial x} < 0, j \in J \setminus \bar{J}. \tag{84}$$

Multiplying both sides of (83) by h^T we get $\sum\limits_{j \in J} h^T \frac{\partial f(x^0,t)}{\partial x}\lambda_j = 0$. From this equality and (84), it follows that $\lambda_j = 0,\ j \in J \setminus \bar{J}$.

For $j \in \bar{J}$, let us consider t_j^* and corresponding problem (74) with $t_j = t_j^*$. It is easy to show that the feasible set of this problem belongs to the set

$$\{\eta \in \mathbb{R}^p : \eta = \sum_{i \in P(j)} b_i(j)\beta_i + \sum_{i \in I(j,x^0)} a_i(j)\alpha_i, \alpha_i \geq 0, i \in I(j, x^0)\},$$

where $I(j, x^0) = \{i \in I(j) : \frac{\partial f^T(x^0,t_j^*)}{\partial t} a_i(j) = 0\}$. It follows from (81), (82) that for any η from this set, the inequality $\xi^T \frac{\partial^2 f(x^0,t_j^*)}{\partial x \partial t}\eta \leq 0$ takes place. Notice that for any η that is feasible in problem (74) we have $\eta^T \frac{\partial^2 L(x^0,t_j^*,y(j))}{\partial t^2}\eta \leq 0$. Consequently, $Q(x^0, \xi, t_j^*, y(j)) = 0$ for $j \in \bar{J}$.

Taking into account these equalities and equalities $\lambda_0 = 0$, $\lambda_j = 0$, $j \in J \setminus \bar{J}$, condition (73) takes the form $\sum\limits_{j \in \bar{J}} \lambda_j \xi^T \frac{\partial^2 f(x^0,t_j^*)}{\partial x^2}\xi > 0$. But this contradicts (80). Obtained contradiction proves that the vector $x^0 \in X$ is an optimal solution of problem (78) where $|J^0| < \infty$.

Taking into account convexity of problem (78) and applying Helly's theorem one can show that the set J^0 can be chosen in a such way that $\{t_j, j \in J^0\} \subset T_a(x^0) \setminus T^*$, $|J^0| \leq n$. The theorem is proved. □

Acknowledgements This work was partially supported by Belarusian State Scientific Program "Convergence" and Portuguese funds through CIDMA - Center for Research and Development in Mathematics and Applications, and Portuguese funds through CIDMA - Center for Research and Development in Mathematics and Applications and FCT - Portuguese Foundation for Science and Technology, within the project UID/MAT/04106/2013.

The authors express their gratitude to the anonymous referee whose remarks and suggestions have permitted to improve the quality of the paper.

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